ADM H-2051 — CH-02 (01) — Dr Satish Kr Gupta.

Partially ordered set (Poset)

A relation R on a set A is said to be partial order if
① R is reflexive $aRa, \forall a \in A$ ② R is antisymmetric aRb, bRa then $a = b$
③ R is transitive $aRb, bRc \Rightarrow aRc$

Note: ① $(A, \leq)$ is called poset
Note: ② "$\leq$" may be R, OR less than or equal to OR $\subseteq$ or g.c.d of a & b & l.c.m of a & b etc

Example of a poset. Let Relation is $a|b$ i.e $a \leq b$ … $a|b$

Let N be a set of natural numbers then
① $(N, \leq)$ is a poset if $\leq$ means less than equal to
② $(N, |)$ is a poset if $a|b, \forall a, b \in N$

③ Let $A = \{a, b, c\}$ then
$P(A) = \{\phi, (a), (b), (c), (ab), (ac), (bc), (abc)\}$
with "$\subseteq$" as relation is a poset.

Comparable elements in a poset.
Let $(A, \leq)$ be a poset then any two elements of A namely $a \in A$ and $b \in R$ are said to be comparable if aRb or bRa. Here relation is "$\leq$"

Non-comparable elements in a poset.
Let $(A, \leq)$ be a poset then $a \in A, b \in A$ are said to be non-comparable if neither $a \leq b$ nor $b \leq a$
i.e neither aRb nor bRa.

R means relation this may be ① $\leq$ ② $\subseteq$ ③ divisibility etc

Example of non-comparable elements

Let $(N, |)$ be a poset

The elements 2 and 5 are not comparable since neither 2 divides 5 nor 5 divides 2. Thus in a poset every pair of elements need not be comparable

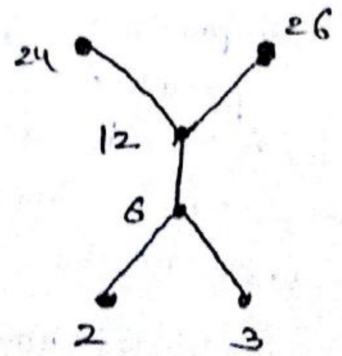
Totally ordered set. Let $(A, \leq)$ be a poset. $(A, \leq)$ is called chain or totally ordered if every two elements in A are comparable

Note. Totally ordered we mean linearly ordered also

Hasse-Diagram Let $(A, \leq)$ be a poset then representation of $(A, \leq)$ by a diagram is called Hasse-diagram. In this we join the comparable elements by a line segment and we do not join non-comparable elements

Example. Let $\{X = \{2, 3, 6, 12, 24, 36\}, |\}$ be a poset

The diagram of it is as

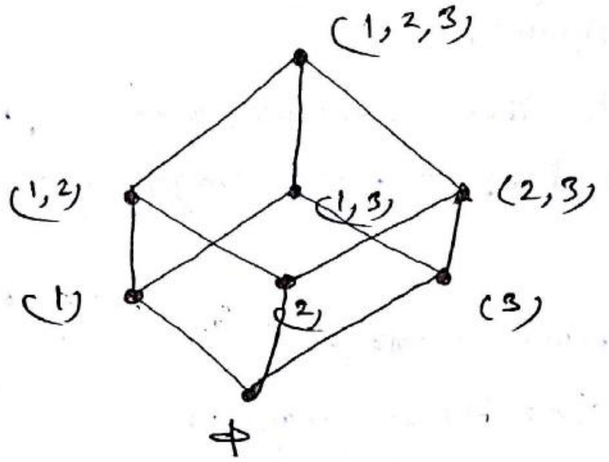


Hasse diagram of $(X, |)$.

Ex. 02. Let $X = \{1, 2, 3\}$

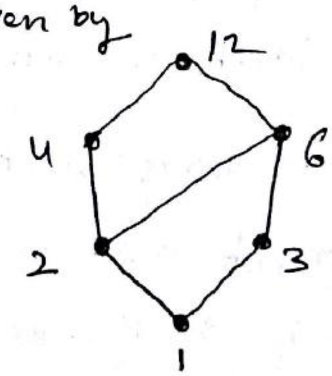
$P(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

then Hasse-diagram of $\{P(X), \subseteq\}$ is



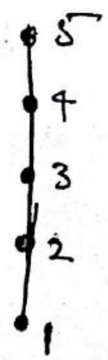
Ex. 03. Let $\{X = \{1, 2, 3, 4, 6, 12\}, |\}$ be a poset

Its Hasse diagram is given by



Ex-04 Let $\{X = \{1, 2, 3, 4, 5\}, \leq\}$ be a poset

Its Hasse-diagram is



Hasse-diagram of $(X, \leq)$.

 Dr Satish Kr Gupta

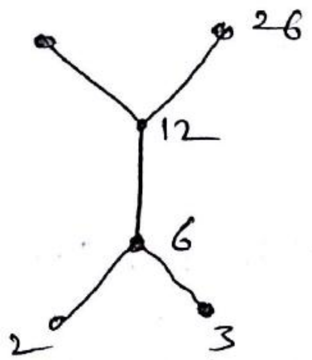
Minimal Elements in a poset.

Let $(P, \leq)$ be a poset. An element $a \in P$ is called minimal element of P if there exists no $b \in P$ such that $b \leq a$ $(b \neq a)$.

Note. ① A minimal element in a poset need not be unique.

② All those elements, which appear at the lowest level of a Hasse diagram are minimum elements.

example
minimal elements in the Hasse-diagram are 2 and 3.



③ Every finite poset has at least one minimal element.

Maximal Elements in a poset.

Let $(P, \leq)$ be a poset. An element $a \in P$ is called maximal element of P if $\exists$ no $b \in P$ such that $b \geq a$ $(b \neq a)$,

Example. In the above Hasse-diagram maximal elements of $\{\{2, 3, 6, 12, 24, 36\}, |\}$, are 24 and 36.

Least elements in a poset. Let $(P, \leq)$ be a poset.

If there exists an element $a \in P$ such that $a \leq x, \forall x \in P$ then a is called least element of P. It is always unique.

Ex ① let $\{X = \{1, 2, 3, 4, 6, 12\}, |\}$ be a poset

Its least element is 1 as $1|2$, $1|3$, $1|4$, $1|6$, $1|12$, i.e. $a \leq x, \forall x \in P$. $\leq$ we mean $|$, divide

② In $\{X = \{2, 3, 6, 12, 24, 36\}, |\}$

There is no least element but minimal elements are 2 and 3.

③ In $\{P(X) = \{\phi, (1), (2), (3), (12), (13), (2,3), (123)\}, X = (1,2,3), \subseteq\}$

the least element is ϕ.

Greatest element in a poset. Let $(P, \leq)$ be a poset.

If there exists an element $a \in P$ such that $a \geq x, \forall x \in P$ then a is called greatest element of P. It is always unique.

Ex. ① let $\{X = \{1, 2, 3, 4, 6, 12\}, |\}$ be a poset

Its greatest element is 12

② In $\{X = \{2, 3, 6, 12, 24, 36\}, |\}$, there is no greatest element but maximal elements are 24 and 36.

③ In $\{P(X) = \{\phi, (1), (2), (3), (12), (13), (2,3), (123)\}, \subseteq\}$ the greatest element is (123).

Upper bound of a subset of a poset

Let $(P, \leq)$ be a poset. Let $A \subseteq P$. An element $x \in P$ is called upper bound of A if $\exists\, a \in A$ such that $a \leq x$, $\forall\, a \in A$.

Ex ① If $X = \{1, 2, 3\}$

$P(X) = \{\phi, (1), (2), (3), (12), (13), (23), (123)\}$

Let $A = \{\phi, (2), (3), (12)\}$

then upper bound of A is $\{1, 2, 3\}$

Here $x = (1\,2\,3)$ & $a \leq x$, $\forall\, a \in A$.

i.e each element of A is a subset of $(1, 2, 3)$.

Let $B = \{(3), (13)\}$

Then upper bounds of B are (13) and (123)

as $a \subseteq (13)$, $\forall\, a \in B$

and $a \subseteq (1, 2, 3)$, $\forall\, a \in B$

② If $\{X = \{2, 3, 6, 12, 24, 36\}, |\}$ be a poset

$A = \{2, 3, 6\}$

then upper bounds of A are 6, 12, 24, 36

i.e $a \leq x$, $\forall\, a \in A$.

x is upper bd of A.

Supremum of A = l.u.b of A in $(P, \leq)$

ex In ② upper bounds of A are 6, 12, 24, 36
& sup of A = 6 [$\because$ l.u.b of A is 6]

CH-02 ADM Dr Satish Kr Gupta

07 H-2051

Lower bound of a subset of a poset

Let $(P, \leq)$ be a poset. Let $A \subseteq P$. Then an element $x \in P$ is called lower bound of A if $\exists\, a \in A$ such that $a \geq x,\ \forall\, a \in A$.

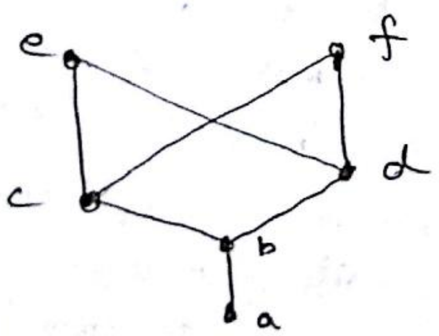
① $P(X) = \{\phi, (1), (2), (3), (12), (13), (23), (123)\}$

$A = \{(12), (13)\}$

Lower bounds of A are $\phi, (1)$,

$\text{Inf } A = \text{g.l.b of } A = (1)$.

Note. If any two Upper bounds of A are not comparable then sup A does not exists



Let $A = \{b, c, d\}$

Upper bounds of A are e and f

and e & f are not comparable elements

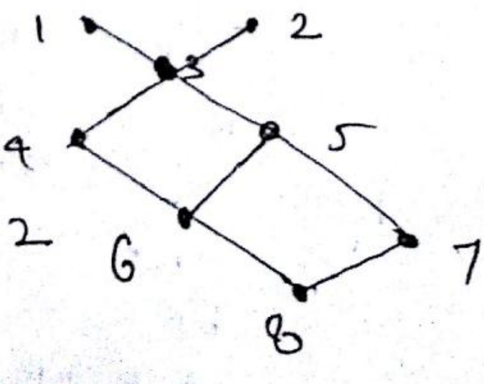
$\Rightarrow$ sup A does not exists.

② $P = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$A = \{4, 5, 7\}$

Upper bound of A are 1, 3, 2

$\rightarrow \sup A = \text{l.u.b of } A = 3$.



Let $(X, \leq)$ i.e (X, R) be a poset then $(X, \bar{R})$ is called dual of (X, R).

Here if R is $\leq$ then $\bar{R}$ we mean "$\geq$".

Theorem. If (X, R) be a poset then prove that $(X, \bar{R})$ is also a poset

i.e Dual of a poset is poset, prove it.

Proof. Let (X, R) be a poset

To prove $(X, \bar{R})$ is a poset

To prove $\bar{R}$ is reflexive.

Given R is reflexive

$\Rightarrow aRa, \forall a \in X$

$\Rightarrow a\bar{R}a, \forall a \in X$

$\Rightarrow \bar{R}$ is reflexive

To prove $\bar{R}$ is anti-symmetric, Let $a\bar{R}b, b\bar{R}a$

it is given R is anti-symmetric

We have $aRb \Rightarrow b\bar{R}a$

$bRa \Rightarrow a\bar{R}b$

$aRb, bRa \Rightarrow a = b$ (given)

$\Rightarrow b\bar{R}a, a\bar{R}b \Rightarrow a = b$

$\therefore \bar{R}$ is anti-symmetric

To prove $\bar{R}$ is transitive

Let $a\bar{R}b \Rightarrow bRa$

$b\bar{R}c \Rightarrow cRb$

Now $cRb, bRa \Rightarrow cRa$

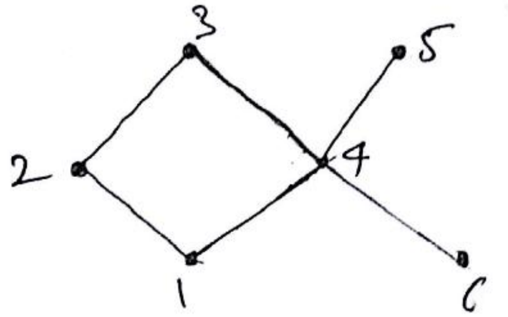
$\Rightarrow a\bar{R}c$

$\Rightarrow \bar{R}$ is transitive Hence dual of a poset is poset.

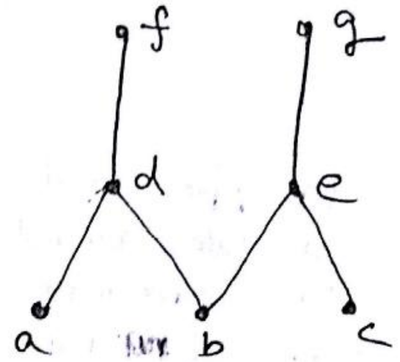
(09) ADM Dr Satish Kr Gupta

CH-02 H-2051

Example (1): Find all the maximal and minimal elements of posets whose Hasse diagrams



(a)



(b)

maximal elements in (a) 3, 5

in (b) f, g

minimal elements in (a) 1, 6

in (b) a, b, c

Example (2) Find the greatest and least elements, if they exist of the following posets

$\{A = \{2, 4, 6, 8, 12, 18, 24, 36, 72\}, |\}$

Ans. greatest element = 72

least element = 2

Example. (3) Let $\{A = \{2, 3, 4, 6, 8, 12, 24, 48\}, |\}$ be a poset. Find

(a) all upper bounds of B if $B = \{4, 6, 12\}$

(b) all lower bounds of B

(c) the least upper bound of B

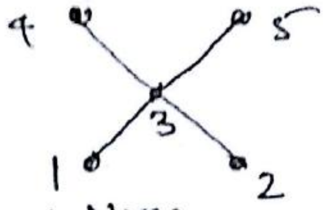
(d) the greatest lower bound of B.

Ans (a) 12, 24, 48 [$\because x \geq a, \forall a \in B$, a will divide x]

(b) 2 [$\because x \leq a, \forall a \in B$] x will divide all $a \in B$

(c) 12 (d) 2

Example 04. Find all upper bounds, all lower bounds, supremum and infimum of the following



Ans (1) Upper bound → None
(2) lower bound → None
(3) supremum → none
(4) infimum → none

Product of two posets

Let $(A, \leq)$, $(B, \leq)$ be any two posets

then $A \times B = \{(a, b) \mid a \in A, b \in B\}$

relation on $A \times B$ is defined as

$(a_1, b_1) \leq (a_2, b_2)$ if $a_1 \leq a_2$ and $b_1 \leq b_2$ on A & B resp.

To show $(A \times B, \leq)$ is also a poset.

(1) To show "$\leq$" is reflexive. let $(a_1, b_1) \in A \times B$

then $(a_1, b_1) R (a_1, b_1) \Rightarrow a_1 \leq a_1,\ b_1 \leq b_1$

$\Rightarrow$ "$\leq$" is reflexive

(2) To show "$\leq$" is transitive. let $(a_1, b_1) \in A \times B$
$(a_2, b_2) \in A \times B$

$(a_1, b_1) \leq (a_2, b_2) \Rightarrow a_1 \leq a_2,\ b_1 \leq b_2$ — (1)

$(a_2, b_2) \leq (a_1, b_1) \Rightarrow a_2 \leq a_1,\ b_2 \leq b_1$ — (2)

(1) & (2) $\Rightarrow a_1 = a_2,\ b_1 = b_2$

(3) To show "$\leq$" is transitive. let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in A \times B$

$(a_1, b_1) \leq (a_2, b_2) \Rightarrow a_1 \leq a_2$ and $b_1 \leq b_2$

$(a_2, b_2) \leq (a_3, b_3) \Rightarrow a_2 \leq a_3$ and $b_2 \leq b_3$
(i) (ii)

(i) $\Rightarrow a_1 \leq a_2 \leq a_3 \Rightarrow a_1 \leq a_3$

(ii) $\Rightarrow b_1 \leq b_2 \leq b_3 \Rightarrow b_1 \leq b_3$

$\therefore (a_1, b_1) \leq (a_2, b_2) \leq (a_3, b_3) \Rightarrow a_1 \leq a_3,\ b_1 \leq b_3$

Hence $(A \times B, \leq)$ is a poset.

CH-02 11 ADM H-2001 Dr Satish Kr Gupta

Isomorphic Posets let (A, R) and (B, R') be any two posets

let $f : A \longrightarrow B$ is called isomorphism of

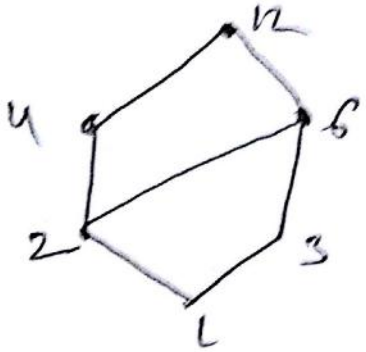
$$x R y \Leftrightarrow f(x) R' f(y), \ \forall x, y \in A$$

ex let $\{A = \{1, 2, 3, 4, 6, 12\}, |\}$ be a poset

$\{B = \{1\ 2\ 3\ 4\ 6\ 12\}, \leq\}$ be another set

let us define a relation

$$x | y \Leftrightarrow f(x) \leq f(y), \ \forall x, y \in A$$



(a)

(b)

In fig (a) 2 & 3 are not comparable, In fig (b) 2 & 3 are comparable

$\Rightarrow \quad (A, |) \not\cong (B, \leq)$

ie (A, R) is not isomorphic to (B, R').

(2) Let $\{A = \{1, 2, 3, 6\}, |\}$

& $B = \{\phi, (a), (b), (ab)\}, \subseteq$ be any two posets

s.t $f(1) = \phi \qquad f(3) = (b)$

$f(2) = (a) \qquad f(6) = (ab)$ from A to B

then f is an isomorphism $f(x) \subseteq f(y), \forall x, y \in A$

ie $x | y \Leftrightarrow f(x) \subseteq f(y)$

Take $x = 2, y = 6 \Rightarrow 2|6 \qquad f(2) = (a), f(6) = (a, b)$

clearly $f(2) \subseteq f(6)$

Similarly we can prove other.

Hence $A \cong B$.